Topology and its Applications 21 (1985) 237-240
North-Holland

BOUNDARY SET IMBEDDINGS IN THE HILBERT CUBE

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Received 4 June 1984
Revised 12 September 1984

We make the following remarks. Every boundary set in the Hilbert cube can be reimbedded as a dense σZ-set whose complement is not l^2. There exist spaces that can be imbedded as a boundary set in more than one way. We also give a necessary condition for a space to be imbeddable as a boundary set.

AMS (MOS) Subj. Class.: 57N20, 54C25, 54G20

Hilbert space   dense σZ-imbedding
boundary set in the Hilbert cube

A boundary set in the Hilbert cube Q is a σZ-set whose complement is homeomorphic to the separable Hilbert space l^2. Curtis has presented with [5] a thorough study of the concept. We wish to place a few remarks. Curtis [5, Example 5.4] shows that there exist dense σZ-set copies of the (f-d) cap sets Σ and σ which are not boundary sets. We give in this paper a simple argument that shows that every boundary set has this property. Curtis also proves that Σ and σ admit essentially only one imbedding as a boundary set in Q and he asks whether this holds for any boundary set. We show that the answer is no. Finally, we prove that every boundary set contains infinite chains of 'strongly linked' compacta. For background information see Bessaga and Pełczyński [2] and Curtis [5].

Proposition 1. *Every boundary set in Q can be reimbedded as a dense σZ-set whose complement is not homeomorphic to l^2.*

Proof. Let I denote the interval $[0, 1]$ and consider the following dense, open subsets of the Hilbert cube $I^2 \times Q$:

$$O_1 = (I^2 \setminus \{0, 0\}) \times Q$$

and

$$O_2 = (I^2 \setminus \{(x, x) \mid x \in [0, \tfrac{1}{2}]\}) \times Q.$$

Note that there exists a homeomorphism $h: O_1 \to O_2$. Let A be a boundary set in $I^2 \times Q$. Since the complement of O_1 is a Z-set we may assume that $A \subset O_1$. According

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to Chapman [3, Theorem 3.1], $h(A)$ is a dense σZ-set in $I^2 \times Q$ because A is a dense σZ-set. It is easily seen that $C = (I^2 \times Q) \backslash O_2$ is not a Z-set in $I^2 \times Q$. Since $h(A)$ is a σZ-set, this implies that C is also not a Z-set in $(I^2 \times Q) \backslash h(A)$. In I^2 every compactum is a Z-set [1], whence $h(A)$ is not a boundary set. □

The proof actually shows that every dense σZ-set in the Hilbert cube can be reimbedded as a dense σZ-set which is not target-dense. (Target-density, which is a necessary condition for a boundary set, is defined as follows: a subset X of Q is target-dense if for every open covering $\mathcal{U}$ of Q there exists a compactum $K \subset X$ such that, for every neighbourhood U of K in Q, there is a map $f: Q \to U$ that is $\mathcal{U}$-close to the identity.)

Remark 2. Curtis [4] has shown that a space can be imbedded as a dense σZ-set in Q iff it is σ-compact and nowhere locally compact. For such spaces X we can define the following concept: Q is *dense homogeneous with respect to* X if for any two dense σZ-set copies X_1 and X_2 of X in Q, there exists a homeomorphism $h: Q \to Q$ with $h(X_1) = h(X_2)$. In [6] it is proved that there are precisely three 0-dimensional spaces with this property. Note that it is easy to construct spaces of any dimension for which Q is dense homogeneous. Simply take the union of a countable dense subset and a Z-set in Q. The following proposition shows, however, that 'nice' higher-dimensional spaces for which Q is dense homogeneous are rare.

Proposition 3. *If A is a dense σZ-set in Q such that for every $x \in Q$ the space $\{x\} \cup A$ is locally connected then Q is not dense homogeneous with respect to A.*

Proof. Consider the sets $O_1 = (0, 1)^2 \times Q$ and

$$O_2 = O_1 \backslash \left(\left\{ \frac{1}{n} \,\middle|\, n \in \mathbb{N} \right\} \times [0, \tfrac{1}{2}] \times Q \right),$$

and construct a homeomorphism $h: O_1 \to O_2$. If A is a dense σZ-set in $I^2 \times Q$ then we may assume that $A \subset O_1$. It follows that $h(A)$ is a dense σZ-set in $I^2 \times Q$. Observe that if $x \in \{(0, 0)\} \times Q$ then $\{x\} \cup O_2$ is not locally connected at x. Since $h(A)$ is dense in O_2 this implies that $\{x\} \cup h(A)$ is not locally connected. □

Remark 4. In [8] we constructed a skeletoid A_k for the collection of $\leqslant k$-dimensional Z-sets in the Hilbert cube. It follows from this proposition that Q is not dense homogeneous w.r.t. A_k for $k > 0$, whereas Curtis and Van Mill [6] have shown that Q is dense homogeneous w.r.t. A_0.

We shall now give a negative answer to Curtis [5, Question 6.6], i.e., we give a space that can be imbedded in Q as (deformation) boundary set in more than one way. Consider the f-d capset σ in Q. Assume that the interval I is a subset of $Q \backslash \sigma$ and select a sequence $(Q_i)_{i=1}^{\infty}$ of disjoint Hilbert cubes in $Q \backslash (\sigma \cup I)$ such that the end points of I are the cluster points of $(Q_i)_{i=1}^{\infty}$. Since $Q\backslash\sigma \approx l^2$ we have that I and Q_i are Z-sets in Q and $A = \sigma \cup \bigcup_{i=1}^{\infty} Q_i$ is a boundary set. Let $\tilde{Q}$ be the space we obtain from Q if we identify I to a point a and let q be the decomposition mapping. Since I is a Z-set with trivial shape we have $\tilde{Q} \approx Q$ and $\tilde{Q}\backslash A \approx l^2$ (one can for instance use [3, Section 25] and [11], but simpler arguments are known). As above we find that A is also a σZ-set in $\tilde{Q}$ (and hence a boundary set). Assume that there is a homeomorphism $h: Q \to \tilde{Q}$ with $h(A) = A$. The set $\bigcup_{i=1}^{\infty} Q_i$ is in A topologically characterized by: $x \in \bigcup_{i=1}^{\infty} Q_i$ iff for every countable union F of finite-dimensional compacta in A the pathwise component of x in $(A\backslash F) \cup \{x\}$ contains more than one point. This can be seen as follows. Let F be such a countable union of f-d compacta. If $x \in Q_i$ then $\{x\} \cup (Q_i\backslash F)$ is pathwise connected, see [9]. If $x \in \sigma$ and P is a path in $(A\backslash\sigma) \cup \{x\}$ containing x, then P is covered by the countable, closed, disjoint collection $\{\{x\}\} \cup \{Q_i \mid i \in \mathbb{N}\}$. Sierpiński's theorem [12] implies that $P = \{x\}$. So we may conclude that

$$h\left(\bigcup_{i=1}^{\infty} Q_i\right) = \bigcup_{i=1}^{\infty} Q_i$$

which contradicts

$$\mathrm{cl}_Q\left(\bigcup_{i=1}^{\infty} Q_i\right)\Big\backslash \bigcup_{i=1}^{\infty} Q_i = \{0, 1\} \quad \text{and} \quad \mathrm{cl}_{\tilde{Q}}\left(\bigcup_{i=1}^{\infty} Q_i\right)\Big\backslash \bigcup_{i=1}^{\infty} Q_i = \{a\}.$$

Every boundary set is a dense σZ-set and hence σ-compact and nowhere locally compact. Curtis [5] gives a number of other intrinsic properties of boundary sets, among them, every nonempty open subset is infinite-dimensional. We present the following stronger result, which is useful for recognizing 'fake topological Hilbert spaces', see [10, Section 5.4].

Theorem 5. *Let A be a boundary set in Q and let O be a nonempty open subset of A. If A is written as a union of compacta $F_1, F_2, F_3, \ldots$, then there is for every $n \in \mathbb{N}$ an infinite set $\{i_m \mid m \in \mathbb{N}\}$ of natural numbers such that $\dim(F_{i_m} \cap F_{i_{m+1}} \cap O) \geq n$ for each $m \in \mathbb{N}$.*

In fact Curtis' result was proved for the weaker concept of a target-dense imbedded set. It is easily seen that the theorem does not hold for these sets.

Proof. Assume that we have a sequence $(F_i)_{i=1}^{\infty}$ and an $n \in \mathbb{N}$. Define the following equivalence relation on $\mathbb{N}$: $m \sim l$ if there is a sequence $m = i_1, i_2, \ldots, i_j = l$ in $\mathbb{N}$ with $\dim(F_{i_r} \cap F_{i_{r+1}} \cap O) \geq n$ for $r = 1, 2, \ldots, j-1$. If there is an infinite equivalence class then we are done. If every class is finite we define new compacta $G_{[i]} = \bigcup \{F_j \mid j \sim i\}$, where $[i]$ is the class to which i belongs. Note that if $[i] \neq [j]$, then $\dim(G_{[i]} \cap G_{[j]} \cap O) < n$. Let U be an open, nonempty subset of $Q\backslash A$ whose closure in Q misses $A\backslash O$. Consider the complete space

$$Z = I^{n+2}\backslash((0, 1)^{n+1} \times \{0\}),$$

i.e. the $(n+2)$-cube minus the interior of an $(n+1)$-face. Since $Q\backslash A \approx l^2$ we may imbed Z as a closed set in $Q\backslash A$ with $Z \subset U$. Let the n-sphere S^n be represented by $\partial I^{n+1} \times \{0\} \subset Z$, where ∂I^{n+1} is the boundary of I^{n+1}. Let $\bar{Z}$ be the closure of Z in Q and put $Z^* = \bar{Z}\backslash Z$. Since $I^{n+1} \times (0, 1]$ is locally compact we have $Z^* \cup S^n$ is compact. Moreover, Z^* is a subset of O. Since Z^* is covered by the compacta $\{\bar{Z} \cap G_{[i]} \mid i \in \mathbb{N}\}$ and $\dim(G_{[i]} \cap G_{[j]} \cap O) < n$ if $[i] \neq [j]$, the generalized Sierpiński theorem [7] implies that S^n is a retract of $Z^* \cup S^n$. Consequently, there is a retraction $f: Z^* \cup (\partial I^{n+1} \times I) \to S^n$ such that $f(x, \varepsilon) = (x, 0)$ for $(x, \varepsilon) \in \partial I^{n+1} \times I$. Since S^n is an ANR we can find an open neighbourhood V of $Z^* \cup (\partial I^{n+1} \times I)$ in $\bar{Z}$ and an $\bar{f}: V \to S^n$ that extends f. Note that $\bar{Z}\backslash V$ is a compact subset of $I^{n+1} \times (0, 1]$ whence there exists an $\varepsilon \in (0, 1]$ with $I^{n+1} \times \{\varepsilon\} \subset V$. This implies that there is a retraction $I^{n+1} \to \partial I^{n+1}$. So we may conclude that not all equivalence classes are finite. □

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